

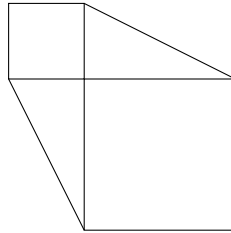
PURPLE COMET! MATH MEET April 2021

HIGH SCHOOL - PROBLEMS

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Problem 1

The diagram shows two intersecting line segments that form some of the sides of two squares with side lengths 3 and 6. Two line segments join vertices of these squares. Find the area of the region enclosed by the squares and segments.

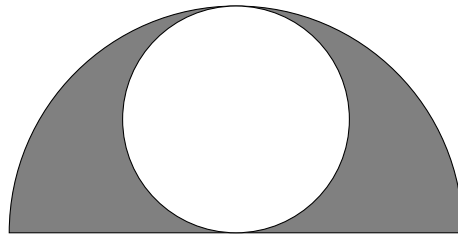


Problem 2

A furniture store set the sticker price of a table 40 percent higher than the wholesale price that the store paid for the table. During a special sale, the table sold for 35 percent less than this sticker price. Find the percent the final sale price was of the original wholesale price of the table.

Problem 3

The diagram shows a semicircle with diameter 20 and the circle with greatest diameter that fits inside the semicircle. The area of the shaded region is $N\pi$, where N is a positive integer. Find N .



Problem 4

A building contractor needs to pay his 108 workers \$200 each. He is carrying 122 one hundred dollar bills and 188 fifty dollar bills. Only 45 workers get paid with two \$100 bills. Find the number of workers who get paid with four \$50 bills.

Problem 5

There were three times as many red candies as blue candies on a table. After Darrel took the same number of red candies and blue candies, there were four times as many red candies as blue candies left on the table. Then after Cloe took 12 red candies and 12 blue candies, there were five times as many red candies as blue candies left on the table. Find the total number of candies that Darrel took.

Problem 6

A rectangular wooden block has a square top and bottom, its volume is 576, and the surface area of its vertical sides is 384. Find the sum of the lengths of all twelve of the edges of the block.

Problem 7

Among the 100 constants $a_1, a_2, a_3, \dots, a_{100}$, there are 39 equal to -1 and 61 equal to $+1$. Find the sum of all the products $a_i a_j$, where $1 \leq i < j \leq 100$.

Problem 8

Pam lists the four smallest positive prime numbers in increasing order. When she divides the positive integer N by the first prime, the remainder is 1. When she divides N by the second prime, the remainder is 2. When she divides N by the third prime, the remainder is 3. When she divides N by the fourth prime, the remainder is 4. Find the least possible value for N .

Problem 9

Find k such that $k\pi$ is the area of the region of points in the plane satisfying

$$\frac{x^2 + y^2 + 1}{11} \leq x \leq \frac{x^2 + y^2 + 1}{7}.$$

Problem 10

A semicircle has diameter \overline{AB} with $AB = 100$. Points C and D lie on the semicircle such that $AC = 28$ and $BD = 60$. Find CD .

Problem 11

There are nonzero real numbers a and b so that the roots of $x^2 + ax + b$ are $3a$ and $3b$. There are relatively prime positive integers m and n so that $a - b = \frac{m}{n}$. Find $m + n$.

Problem 12

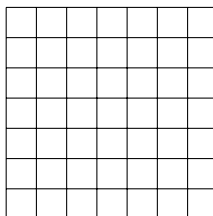
Let L_1 and L_2 be perpendicular lines, and let F be a point at a distance 18 from line L_1 and a distance 25 from line L_2 . There are two distinct points, P and Q , that are each equidistant from F , from line L_1 , and from line L_2 . Find the area of $\triangle FPQ$.

Problem 13

Two infinite geometric series have the same sum. The first term of the first series is 1, and the first term of the second series is 4. The fifth terms of the two series are equal. The sum of each series can be written as $m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Problem 14

Each of the cells of a 7×7 grid is painted with a color chosen randomly and independently from a set of N fixed colors. Call an edge *hidden* if it is shared by two adjacent cells in the grid that are painted the same color. Determine the least N such that the expected number of hidden edges is less than 3.



Problem 15

Find the value of x where the graph of

$$y = \log_3 \left(\sqrt{x^2 + 729} + x \right) - 2 \log_3 \left(\sqrt{x^2 + 729} - x \right)$$

crosses the x -axis.

Problem 16

Paula rolls three standard fair dice. The probability that the three numbers rolled on the dice are the side lengths of a triangle with positive area is $\frac{m}{n}$, where m and n are relatively prime positive integers.

Find $m + n$.

Problem 17

For real numbers x let

$$f(x) = \frac{4^x}{25^{x+1}} + \frac{5^x}{2^{x+1}}.$$

Then $f\left(\frac{1}{1 - \log_{10} 4}\right) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 18

The side lengths of a scalene triangle are the roots of the polynomial

$$x^3 - 20x^2 + 131x - 281.3.$$

Find the square of the area of the triangle.

Problem 19

Let a, b, c, d be an increasing arithmetic sequence of positive real numbers with common difference $\sqrt{2}$. Given that the product $abcd = 2021$, d can be written as $\frac{m+\sqrt{n}}{\sqrt{p}}$, where m, n , and p are positive integers not divisible by the square of any prime. Find $m + n + p$.

Problem 20

Let $ABCD$ be a convex quadrilateral with positive integer side lengths, $\angle A = \angle B = 120^\circ$, $|AD - BC| = 42$, and $CD = 98$. Find the maximum possible value of AB .

Problem 21

Let a, b , and c be real numbers satisfying the equations

$$a^3 + abc = 26$$

$$b^3 + abc = 78$$

$$c^3 - abc = 104.$$

Find $a^3 + b^3 + c^3$.

Problem 22

The least positive angle α for which

$$\left(\frac{3}{4} - \sin^2(\alpha)\right) \left(\frac{3}{4} - \sin^2(3\alpha)\right) \left(\frac{3}{4} - \sin^2(3^2\alpha)\right) \left(\frac{3}{4} - \sin^2(3^3\alpha)\right) = \frac{1}{256}$$

has a degree measure of $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 23

The sum

$$\sum_{k=3}^{\infty} \frac{1}{k(k^4 - 5k^2 + 4)^2}$$

is equal to $\frac{m^2}{2n^2}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 24

Let x be a real number such that

$$4^{2x} + 2^{-x} + 1 = (129 + 8\sqrt{2})(4^x + 2^{-x} - 2^x).$$

Find $10x$.

Problem 25

The area of the triangle whose altitudes have lengths 36.4, 39, and 42 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 26

The product

$$\left(\frac{1}{2^3-1} + \frac{1}{2}\right) \left(\frac{1}{3^3-1} + \frac{1}{2}\right) \left(\frac{1}{4^3-1} + \frac{1}{2}\right) \cdots \left(\frac{1}{100^3-1} + \frac{1}{2}\right)$$

can be written as $\frac{r}{s2^t}$, where r , s , and t are positive integers and r and s are odd and relatively prime.

Find $r + s + t$.

Problem 27

Let $ABCD$ be a cyclic quadrilateral with $AB = 5$, $BC = 10$, $CD = 11$, and $DA = 14$. The value of

$AC + BD$ can be written as $\frac{n}{\sqrt{pq}}$, where n is a positive integer and p and q are distinct primes.

Find $n + p + q$.

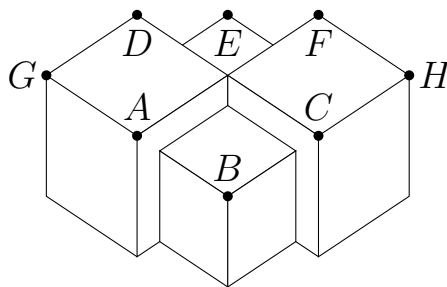
Problem 28

Let $z_1, z_2, z_3, \dots, z_{2021}$ be the roots of the polynomial $z^{2021} + z - 1$. Evaluate

$$\frac{z_1^3}{z_1 + 1} + \frac{z_2^3}{z_2 + 1} + \frac{z_3^3}{z_3 + 1} + \cdots + \frac{z_{2021}^3}{z_{2021} + 1}.$$

Problem 29

Two cubes with edge length 3 and two cubes with edge length 4 sit on plane P so that the four cubes share a vertex, and the two larger cubes share no faces with each other as shown below. The cube vertices that do not touch P or any of the other cubes are labeled A, B, C, D, E, F, G , and H . The four cubes lie inside a right rectangular pyramid whose base is on P and whose triangular sides touch the labeled vertices with one side containing vertices A, B , and C , another side containing vertices D, E , and F , and the two other sides each contain one of G and H . Find the volume of the pyramid.



Problem 30

For positive integer k , define $x_k = 3k + \sqrt{k^2 - 1} - 2(\sqrt{k^2 - k} + \sqrt{k^2 + k})$. Then

$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_{1681}} = \sqrt{m} - n$, where m and n are positive integers. Find $m + n$.